Big Math Ideas

Solving Exponential and Logarithmic Equations

Chapter 6.6

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Chapter 6.6 Solving Exponential and Logarithmic Equations

Core Vocabulary:

Exponential equations: Equations in which variable expressions occur as exponents.

Core Concept:

Property of Equality for Exponential Equations

Algebra: If b is a positive real number other than 1, then b^x = b^y if and only if x=y.
Example: If 3^x = 3^5, then x=5.

Useful for solving an exponential equation when each side of the equation uses the same base (or can be written to use the same base).

If it's not convenient to write each side of an exponential equation using the same base, you can try to solve the equation by taking a logarithm of each side.

Example 1- Solving Exponential Equations
Solve each equation:

a. $100^x = (\frac{1}{10})^{x-3}$ b. $2^x = 7$

Solution

a.

$100^x = (\frac{1}{10})^{x-3}$	Write original equation
$(10^2)^x = (10^{-1})^{x-3}$	Rewrite 100 and $\frac{1}{10}$ as powers with base 10.
$10^{2x} = 10^{-x+3}$	Power of a Power Property
$2x = -x+3$	Property of Equality for Exponential Equations
$x = 1$	Solve for x

b.

$2^x = 7$	Write original equation
$\log_2 2^x = \log_2 7$	Take $\log_2$ *of each side*
$x = \log_2 7$	$\log_b b^x = x$
$x \approx 2.807$	Use Calculator

Questions:

1. In example 1a, what do you observe about the two bases?
2. Why can't example 1b be solved the same way?
3. How do you evaluate $\log_2 7$?

Answers:

1. They can be expressed as powers of 10.
2. It's not possible to write 2 and 7 as powers of the same base.
3. Use the change of base formula

An important application of exponential equations is Newton's Law of Cooling. This law states that for a cooling substance with initial temperatureT_0, the temperature T after t minutes can be modeled by,$T = (T_0 - T_R)e^{-rt} + T_R$ where T_R is the surrounding temperature and r is the cooling rate of the substance.

Example 2- Solving a Real- Life problem
You are cooking stew. When you take it off the stove, its temperature is 212˚F. The room temperature is 70˚F, and the cooling rate of the stew is r = 0.046. How long will it take to cool the stew to a serving temperature of 100˚F? (Use Newton's Law of Cooling).

Solution:

$T = 100,\ T_0 = 212, T_R = 70\ and\ r = 0.046$

$T = (T_0 - T_R)e^{-rt} + T_R$	Newton's Law of Cooling
$100 = (212 - 70)e^{-0.046t} + 70$	Substitute for $T, T_0, T_R, and\ r$
$30 = 142e^{-0.046t}$	Subtract 70 from each side.
$0.211 \approx e^{-0.046t}$	Divide each side by 142.
$\ln 0.211 \approx \ln e^{-0.046t}$	Take natural log of each side.
$-1.556 \approx -0.046t$	$\ln e^x = \log_e e^x = x$
$33.8 \approx t$	Divide each side by -0.046.

Answer: You should wait about 34 minutes before serving the stew.

See example 1 for help

1. $7^{3x+5} = 7^{1-x}$
2. $5^{x-3} = 25^{x-5}$
3. $512^{5x-1} = (\frac{1}{8})^{-4-x}$
4. $3e^{4x} + 9 = 15$
5. $2e^{2x} - 7 = 5$
6. $49^{5x+2} = (\frac{1}{7})^{11-x}$

7. You cook a turkey until the internal temperature reaches 180 degrees. The turkey is placed on the table and the internal temperature reaches 100 degrees and it can be carved. When the room temperature is 72 degrees, the cooling rate of the turkey is r = 0.067. How long do you have to wait to carve the turkey?

8. Suppose you cool a pot of soup in a 75 degrees Fahrenheit room. Right when you take the soup off the stove, you measure its temperature to be 220 degrees Fahrenheit. Suppose after 20 minutes the soup has cooled to 170 degrees Fahrenheit.

a. Find the value of k to the nearest ten-thousandth
b. What will be the temperature of the soup in 30 minutes, to the nearest degree?
c. Suppose you can eat the soup when it is 130 degrees Fahrenheit. How long will it take to cool to this temperature, to the nearest minute?

Use Newton's Law of Cooling

$$T(t) = (T_0 - T_R)e^{kt} + T_R$$

Core Vocabulary:

Logarithmic Equations: Equations that involve logarithms of variable expressions.

Core Concept:

Property of Equality for Logarithmic Equations

Algebra: If b, x and y are positive real numbers with $b \neq 1$, then $\log_b x = \log_b y$ if and only if x=y.
Example: If $\log_2 x = \log_2 7$, then x=7. If x=7, then $\log_2 x = \log_2 7$.

Implies that if you are given an equation x=y, then you can exponentiate each side to obtain an equation of the form $b^x = b^y$.

Example 3- Solving Logarithmic equations

Solve

(a) $\ln(4x-7) = \ln(x+5)$ (b) $\log_2(5x-17) = 3$

Solution:

a.	$\ln(4x-7) = \ln(x+5)$	Write original equation
	$4x - 7 = x + 5$	Property of Equality for Logarithmic equations
	$3x - 7 = 5$	Subtract x from each side.
	$3x = 12$	Add 7 to each side
	$x = 4$	Divide by 3
b.	$\log_2(5x-17) = 3$	Write original equation
	$2^{\log_2(5x-17)} = 2^3$	Exponentiate each side using base 2
	$5x - 17 = 8$	$b^{\log_b x} = x$
	$5x = 25$	Add 17 to each side
	$x = 5$	Divide each side by 5

Example 4- Solving a Logarithmic Equation
Solve: $\log 2x + \log(x-5) = 2$

Solution:

$\log 2x + \log(x-5) = 2$	Write original
$\log[2x(x-5)] = 2$	Product Property of Logarithms
$10^{\log[2x(x-5)]} = 10^2$	Exponentiate each side using base 10
$2x(x-5) = 100$	$b^{\log_b x} = x$
$2x^2 - 10x = 100$	Distributive Property
$2x^2 - 10x - 100 = 0$	Write in standard form
$x^2 - 5x - 50 = 0$	Divide each side by 2
$(x+5)(x+10) = 0$	Factor
$x = 10 \; or \; x = -5$	Zero-Product property

The apparent solution x=-5 is extraneous. The only solution is x=10.

See example 3 for help	See example 4 for help
9. $\ln(4x-7) = \ln(x+11)$	14.$\log_2 x + \log_2(x-2) = 3$
10.$\ln(2x-4) = \ln(x+6)$	15.$\ln x + \ln(x+3) = 4$
11.$\log_2(3x-4) = \log_2 5$	16.$\ln x + \ln(x-2) = 5$
12.$\log_2(4x+8) = 5$	17.$\log_3(x-9) + \log_3(x-3) = 2$
13.$\log_3(x^2+9x+27) = 2$	18.$\log_5(x+4) + \log_5(x+1) = 2$

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Solving Exponential and Logarithmic Equations

Answer Key

1. $7^{3x+5} = 7^{1-x}$

$3x + 5 = 1 - x$
$+x \quad -5 \quad -5 \quad +x$

$\frac{4x}{4} = \frac{-4}{4}$

$\boxed{x = -1}$

2. $5^{x-3} = 25^{x-5}$

$5^{x-3} = 5^{2(x-5)}$

$x - 3 = 2x - 10$
$-x + 10 \quad -x + 10$

$\boxed{7 = x}$

3. $512^{5x-1} = \left(\frac{1}{8}\right)^{-4-x}$

$2^{9(5x-1)} = 2^{-3(-4-x)}$

$2^{45x-9} = 2^{12+3x}$

$45x - 9 = 12 + 3x$
$-3x + 9 \quad +9 \quad -3x$

$\frac{42x}{42} = \frac{21}{42}$

$\boxed{x = \frac{1}{2}}$

4. $3e^{4x} + 9 = 15$
$-9 \quad -9$

$\frac{3e^{4x}}{3} = \frac{6}{3}$

$e^{4x} = 2$

Take natural log of both sides

$\frac{4x}{4} = \frac{\ln(2)}{4}$

$\boxed{x = \frac{\ln(2)}{4} \text{ or } \approx 0.173}$

5. $2e^{2x} - 7 = 5$
$+7 \quad +7$

$\frac{2e^{2x}}{2} = \frac{12}{2}$

$e^{2x} = 6$

$\frac{2x}{2} = \frac{\ln(6)}{2}$

$\boxed{x = \frac{\ln(6)}{2} \text{ or } \approx 0.896}$

6. $49^{5x+2} = \left(\frac{1}{7}\right)^{11-x}$

$7^{2(5x+2)} = 7^{-1(11-x)}$

$7^{10x+4} = 7^{-11+x}$

$10x + 4 = -11 + x$
$-10x + 11 \quad +11 \quad -10x$

$\frac{15}{-9} = \frac{-9x}{-9}$

$\boxed{-\frac{5}{3} = x}$

7. $T(t) = (T_0 - T_R)e^{-rt} + T_R$

$100 = (180 - 72)e^{-rt} + 72$
$-72 \qquad -72$

$28 = 108e^{-rt}$

$\frac{28}{108} = \frac{108e^{-0.067t}}{108}$

$0.259259 = e^{-0.067t}$

$\frac{\ln 0.259259}{-0.067} = \frac{-0.067t}{-0.067}$

$t \approx 20$ minutes

T_0 = inital temperature
T_R = Surrounding temperature
t = inital time in hours
T(t) = temperature after t hours
r = decay constant

8 a. $170 = 75 + (220 - 75)e^{K(20)}$
$-75 \qquad -75$

$\frac{95}{145} = \frac{145e^{K20}}{145}$

$0.65517 = e^{K20}$

$\frac{\ln .65517}{20} = \frac{K20}{20}$ → $K = -0.0211$

b. $T = 75 + (220 - 75)e^{-.0211(30)}$

$T = 75 + 145e^{-0.633}$

$T = 75 + 145 \cdot 0.5309964199$

$T = 151.99$ → $152°$

c. $130 = 75 + (145)e^{-.0211(t)}$
$-75 \qquad -75$

$\frac{55}{145} = \frac{145e^{-0.0211t}}{145}$

$.3793 = e^{-0.0211t}$

$\frac{\ln .3793}{-0.0211} = \frac{-0.0211t}{-0.0211}$ → 45.94 → 46 minutes

9. $\ln(4x-7)=\ln(x+11)$

$4x-7=x+11$

$-x+7 \quad -x+7$

$3x=18$

$\boxed{x=6}$

10. $\ln(2x-4)=\ln(x+6)$

$2x-4=x+6$

$-x+4 \quad -x+4$

$\boxed{x=10}$

11. $\log_2(3x-4)=\log_2 5$

$3x-4=5$

$+4 \quad +4$

$3x=9$

$\boxed{x=3}$

12. $\log_2(4x+8)=5$

$4x+8=2^5$

$4x+8=32$

$4x=24$

$\boxed{x=6}$

13. $\log_3(x^2+9x+27)=2$

$x^2+9x+27=3^2$

$x^2+9x+27=9$

$-9 \quad -9$

$x^2+9x+18=0$

Quadratic Formula

$x=\frac{-9\pm\sqrt{9^2-4\cdot1\cdot18}}{2\cdot1}$

$x=\frac{-9\pm\sqrt{9}}{2} \rightarrow x=\frac{-9\pm3}{2}$

$x=\frac{-9+3}{2}$ $\quad x=\frac{-9-3}{2}$

$\boxed{x=-3 \quad x=-6}$

14. $\log_2 x+\log_2(x-2)=3$

$\log_2 x(x-2)=3$

$\log_2(x^2-2x)=3$

$x^2-2x=2^3 \rightarrow x^2-2x=8$

$x^2-2x-8=0$

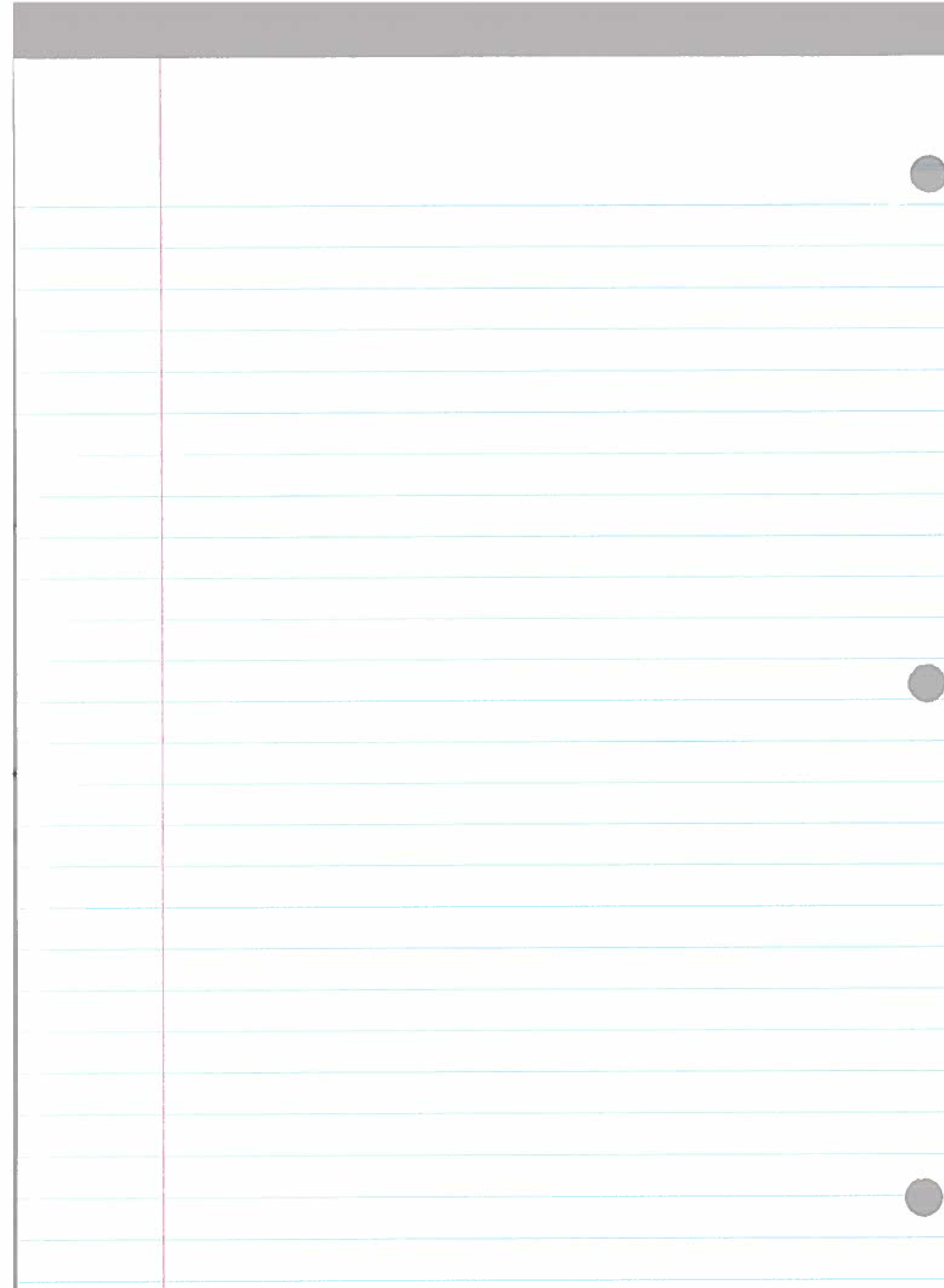
$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-8)}}{2(1)}$

$x=\frac{2\pm\sqrt{36}}{2} \rightarrow x=\frac{2\pm6}{2}$

$x=\frac{2+6}{2}$ $\quad x=\frac{2-6}{2}$

$x=4$ $\quad x\neq-2$ rej

$\boxed{x=4}$



15. $\ln x + \ln(x+3) = 4$

$\ln(x(x+3)) = 4$

$\ln(x^2+3x) = 4$

$x^2+3x = e^4$

$x^2+3x-e^4=0$

$x = \frac{-3 \pm \sqrt{3^2-4(1)(-e^4)}}{2(1)}$

$x = \frac{-3 \pm \sqrt{9+4e^4}}{2}$

$x = \frac{-3+\sqrt{9+4e^4}}{2}$ $x = \frac{-3-\sqrt{9+4e^4}}{2}$ rej

$x = \frac{-3+\sqrt{9+4e^4}}{2}$

$\boxed{x = 6.03977}$

16. $\ln x + \ln(x-2) = 5$

$\ln(x(x-2)) = 5$

$\ln(x^2-2x) = 5$

$x^2-2x = e^5 \rightarrow x^2-2x-e^5=0$

$x = \frac{-(-2) \pm \sqrt{(-2)^2-4\cdot 1\cdot -e^5}}{2(1)}$

$x = \frac{2 \pm \sqrt{4+4e^5}}{2}$

$x = \frac{2+\sqrt{4+4e^5}}{2}$ $x = \frac{2-\sqrt{4+4e^5}}{2}$ rej

$\boxed{x = 13.2235}$

17. $\log_3(x-9)+\log_3(x-3)=2$

$\log_3((x-9)\cdot(x-3))=2$

$\log_3(x^2-3x-9x+27)=2$

$x^2-3x-9x+27=3^2$

$x^2-12x+27=9$

$-9 \quad -9$

$x^2-12x+18=0$

$x = \frac{-(-12) \pm \sqrt{(-12)^2-4\cdot 1\cdot 18}}{2\cdot 1}$

$x = \frac{12 \pm \sqrt{72}}{2} \rightarrow x = \frac{12 \pm 6\sqrt{2}}{2}$

$x = \frac{12+6\sqrt{2}}{2}$ $x = \frac{12-6\sqrt{2}}{2}$

$x = 6+3\sqrt{2}$ $x = 6-3\sqrt{2}$

$\boxed{x = 10.2426}$

18. $\log_5(x+4)+\log_5(x+1)=2$

$\log_5((x+4)\cdot(x+1))=2$

$\log_5(x^2+x+4x+4)=2$

$x^2+5x+4=5^2$

$x^2+5x+4=25 \rightarrow x^2+5x-21=0$

$x = \frac{-5 \pm \sqrt{5^2-4(1)(-21)}}{2(1)}$

$x = \frac{-5 \pm \sqrt{109}}{2}$

$x = \frac{-5+\sqrt{109}}{2}$ $x = \frac{-5-\sqrt{109}}{2}$

$\boxed{x = 2.72015}$