Katelyn Cecer

4.6: The Fundamental Theorem of Algebra

Core Concept ①: Fundamental Theorem of Algebra

Equation	Degree	# of Solutions
$2x-1=0$	1	1
$x^2+x+1=0$	2	2
$x^3+2x^2+x=0$	3	3
$x^4+3x^2+2x=0$	4	4

The largest exponent value represents the number of solutions to the equation

Core Concept ②: Complex Conjugates Theorem

If f is a polynomial function with real coefficients, and $a+bi$ is an imaginary zero, then $a-bi$ is also a zero.

EXAMPLE: Write a polynomial function of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 2 and 3+i. → Because 3+i is a zero, 3-i is also a zero

① $f(x) = (x-2)[x-(3+i)][x-(3-i)]$ (Write f(x) in factored form)

② $= (x-2)[(x-3)-i][(x-3)+i]$ (Regroup terms)

③ $= (x-2)[(x-3)^2-i^2]$ (Multiply)

④ $= (x-2)[(x^2-6x+9)-(-1)]$ (Expand & Substitute for i)

⑤ $= (x-2)(x^2-6x+10)$ (Simplify)

⑥ $= x^3-6x^2+10x-2x^2+12x-20$ (Multiply)

⑦ $= x^3-8x^2+22x-20$ (combine like terms)

Core Concept ③: Descarte's Rule of Signs

- The number of positive real zeros of f is equal to the number of sign changes of coefficients of $f(x)$ or is less than this by an even number (2)

The number of negative real zeros of f is equal to the number of sign changes of the coefficients of $f(-x)$ or is less than this by an even number

EXAMPLE: $x^6-2x^5+3x^4-10x^3-6x^2-8x-8$

① $x^6-2x^5+3x^4-10x^3-6x^2-8x-8$
1 2 3 → 3 positive zeros

② 3-2=1 → 3 & 1 are positive zeros

③ $x^6-2x^5+3x^4-10x^3-6x^2-8x-8$
$(-x)^6-2(-x)^5+3(-x)^4-10(-x)^3-6(-x)^2-8(-x)-8$
$x^6+2x^5+3x^4+10x^3-6x^2+8x-8$
1 2 3 → 3 negative zeros

④ 3-2=1 → 3 & 1 are negative zeros

Posit. Real Zeros	Neg. Real Zeros	Imaginary Zeros	Total Zeros
3	3	0	6
3	1	2	6
1	3	2	6
1	1	4	6

Pos. Zeros: # of sign changes in f(x) & the # of sign changes -2 until a negative number is reached

Neg. Zeros: # of sign changes in f(-x) & the # of sign changes -2 until a negative number is reached

Total Zeros: the value that is the greatest exponent

Imagin. Zeros: Total Zeros - (Posit Real Zeros + Neg Real Zeros)

SUMMARY:

- Fundamental Theorem of Algebra:
  > the largest exponent value represents the number of solutions to the equation
- Complex Conjugates Theorem:
  > if a+bi is a zero, then a-bi is also a zero
- Descarte's Rule of Signs:
  > # of positive zeros = # of sign changes in f(x) or is less than this by 2
  > # of negative zeros = # of sign changes in f(-x) or is less than this by 2
  > imaginary zeros = total zeros - (pos zeros + neg zeros)

PRACTICE: Complete the following exercises on a separate sheet of paper & check answers

1. Identify the number of solutions.

   a) $x^4 + 2x^3 - 4x^2 + x = 0$ ___

   b) $9t^6 - 14t^3 + 4t - 1 = 0$ ___

   c) $g(s) = 4s^5 - s^3 + 2s^7 - 2$ ___

2. Write a polynomial function of least degree that has rational coefficients, a leading coefficient of 1, and the given zeros.

   a) $-5, -1, 2$

   b) $-2, 1, 3$

   c) $2, 5 - i$

   d) $4, -\sqrt{5}$

   e) $3, 4 + i$

3. Describe and correct the error in writing a polynomial function with rational coefficients and the given zeros.

   Zeros: $2, 1 + i$

   $$\begin{aligned} f(x) &= (x-2)[x-(1+i)] \\ &= x(x-1-i) - 2(x-1-i) \\ &= x^2 - x - ix - 2x + 2 + 2i \\ &= x^2 - (3+i)x + (2+2i) \end{aligned}$$

4. Two zeros of $f(x) = x^3 - 6x^2 - 16x + 96$ are 4 and -4. Explain why the third zero must also be a real number.

5. Determine the possible numbers of positive real zeros, negative real zeros, and imaginary zeros for the function.

   a) $g(x) = x^4 - x^2 - 6$

   b) $g(x) = x^3 - 4x^2 + 8x + 7$

   c) $g(x) = x^5 - 2x^3 - x^2 + 6$

   d) $g(x) = x^7 + 4x^4 - 10x + 25$

ANSWER KEY:

1. a) 4 (largest exponent is 4)
   b) 6 (largest exponent is 6)
   c) 7 (largest exponent is 7)

2. a) $f(x) = x^3 + 4x^2 - 7x - 10$

$(x - -5)(x - -1)(x-2)$

$(x+5)(x+1)(x-2)$

distribute

b) $f(x) = x^3 - 2x^2 - 5x + 6$

$(x - -2)(x-1)(x-3)$

$(x+2)(x-1)(x-3)$

distribute

c) $f(x) = x^3 - 12x^2 + 46x - 52$

$(x-2)(x-(5-i))(x-(5+i))$

$(x-2)((x-5)-i)((x-5)+i)$

$(x-2)((x-5)^2 - i^2)$

$(x-2)((x^2-10x+25)-(-1))$

$(x-2)(x^2-10x+26)$

distribute

d) $f(x) = x^3 - 4x^2 - 5x + 20$

$(x-4)(x - -i\sqrt{5})(x - i\sqrt{5})$

$(x-4)(x+i\sqrt{5})(x-i\sqrt{5})$

$(x-4)(x^2 - 5i^2)$

$(x-4)(x^2 - 5(-1))$

distribute

e) $f(x) = x^3 - \cancel{4}11x^2 + 41x - 51$

$(x-3)(x-(4+i))(x-(4-i))$

$(x-3)((x-4)+i)((x-4)-i)$

$(x-3)((x-4)^2 - i^2)$

$(x-3)(x^2-8x+16-(-1))$

$(x-3)(x^2-8x+17)$

distribute

3. The conjugate of the given imaginary zeros was not included

$$f(x) = (x-2)[x-(1+i)][x-(1-i)]$$
$$= (x-2)[(x-1)-i][(x-1)+i]$$
$$= (x-2)[(x-1)^2-i^2]$$
$$= (x-2)[(x^2-2x+1)-(-1)]$$
$$= (x-2)(x^2-2x+2)$$
$$= x^3-2x^2+2x-2x^2+4x-4$$
$$= x^3-4x^2+6x-4$$

4. The function has a degree of 3, so it has 3 solutions. Because the imaginary solutions come in conjugate pairs, there must be an even number of imaginary solutions. Therefore, given that the first two are real, the third must also be real.

5. a) $g(x) = x^4 - x^2 - 6$

$g(-x) = (-x)^4 - (-x)^2 - 6$

$x^4 - x^2 - 6$

Pos.	Neg.	Imag.	Total
1	1	2	4

b) $g(x) = x^3 - 4x^2 + 8x + 7$

$g(-x) = (-x)^3 - 4(-x)^2 + 8(-x) + 7$

$-x^3 - 4x^2 - 8x + 7$

Pos.	Neg.	Imag.	Total
2	1	0	3
0	1	2	3

c) $g(x) = x^5 - 2x^3 - x^2 + 6$

$g(-x) = (-x)^5 - 2(-x)^3 - (-x)^2 + 6$

$-x^5 + 2x^3 - x^2 + 6$

Pos.	Neg.	Imag.	Total
2	3	0	5
2	1	2	5
0	3	2	5
0	1	4	5

d) $g(x) = x^7 + 4x^4 - 10x + 25$

$g(-x) = (-x)^7 + 4(-x)^4 - 10(-x) + 25$

$-x^7 + 4x^4 + 10x + 25$

Pos.	Neg.	Imag.	Total
1	1	5	7