4.5 Solving Polynomial Equations

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Review Guide

Key Points

- Solutions of Polynomial Equations
- Zeros of Polynomial Functions
- The Rational Root Theorem
- Irrational Conjugates Theorem

Solving Polynomial Functions

Polynomial Functions that are factorable may be solved using the Zero-Product Property (Ex. 1)

If the function isn't factorable, the Quadratic Theorem may be used

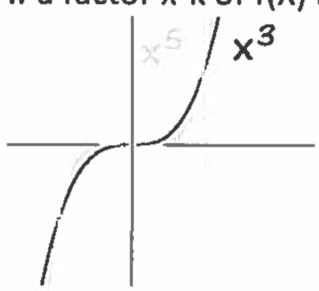
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$f(x)= ax^2+ bx + c$

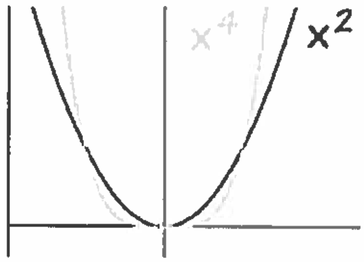
If there is a repeated solution, that root has a multiplicity

Ex. If the factor x-3 appears twice, the root x=3 has a multiplicity of 2

If a factor x-k of f(X) is raised to odd power, the graph of f crosses the x-axis at x=k



If a factor x-k of f(X) is raised to an even power, the graph of f touches the x-axis but doesn't cross at x=k



To find the zeros of a polynomial function, make the function equal to zero and solve by factoring or another method (Ex.2)

The number of zeros for a function is always equal to its degree

Check the graph by plugging it into the graphing calculator

Rational Root Theorem

It lists the possible RATIONAL solutions of the polynomial functions

To find the actual solutions you have to test the possible solutions

To test solutions, use synthetic division or direct substitution

Used when polynomial functions aren't easily factorable

When the leading coefficient isn't 1, the rational solutions increase drastically

This method can be used to find factors and zeros

Formula: P/Q $f(x)= a_nx^n + \ldots + a_1x + a_0$

P- factor of constant term a_0 Q- factor of leading coefficient a_n

Plus minus (±) for everything

(Ex.3)

Irrational Conjugates Theorem

Irrational zeros are conjugates of the form $a + \sqrt{b}$ and $a - \sqrt{b}$

By knowing this theorem, you can write functions from the given zeros to write the product (Ex.4)

Practice Questions

For Questions 1-3 solve the polynomial equation

1. $a^3 - 4a^2 + 4a = 0$

2. $5w^3 = 50w$

3. $v^3 - 2v^2 - 16v = -32$

For Questions 4-5 find the zeros of the polynomial functions and sketch a graph of the function

4. $x^4 + x^3 - 6x^2$

5. $x^6 - 11x^5 + 30x^4$

For questions 6-8 find all the <u>real</u> solutions of the equation

6. $x^3 - 2x^2 - 5x + 6 = 0$

7. $x^3 + 4x^2 - 11x - 30 = 0$

8. $2x^3 - 3x^2 - 50x - 24 = 0$

For questions 9-10 write a polynomial function of the least degree that has a leading coefficient of 1 and the given zeros.

9. -2, 3, 6

10. 4, $6 - \sqrt{7}$

11. Write a polynomial function f of least degree that has rational coefficients, a leading coefficient of 1, and the zeros 4 and $1 - \sqrt{5}$

4.5 Lesson Example Page

Example #1 - Solving a Polynomial Equation by Factoring

$2x^3 - 12x^2 + 18x = 0$

1. $2x(x^2 - 6x + 9) = 0$ — 1. Factor common monomial
2. $2x(x-3)(x-3) = 0$ — 2. Factor further → perfect square pattern

   $2x(x-3)^2 = 0$
3. $\frac{2x}{2} = \frac{0}{2}$, $x = 0$ $\qquad$ $x - 3 = 0$, $+3 \; +3$, $x = 3$ to the multiplicity of 2 — 3. Zero Product Property → find x

The solutions are $x = 0$, $x = 3$ to the mult. of 2.

Example #2 - Finding Zeros of a Polynomial Function

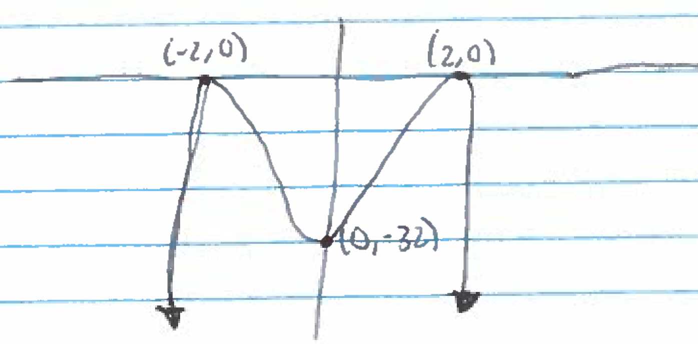
$f(x) = -2x^4 + 16x^2 - 32$ ← y-intercept

1. $0 = \frac{-2x^4 + 16x^2 - 32}{-2}$ — 1. Set f(x) equal to 0
2. — 2. Factor out the GCF (-2)

   $0 = -2(x^4 - 8x^2 + 16)$ — 3. Factor the trinomial further (quadratic form
3. $0 = -2(x^2 - 4)(x^2 - 4)$ — 4. Continue factoring, Difference of 2 squares
4. $0 = -2(x+2)(x-2)(x+2)(x-2)$ — 5. Simplify for multiplicity
5. $0 = -2(x+2)^2(x-2)^2$ $\qquad$ zeros: $x = 2$, $x = -2$

*Because (x+2) and (x-2) are raised to an even power, the graph f touches the x-axis. Y-intercept = -32, (c in the equation abo

Graph Sketch



4.5 Lesson Example Page

Example 3 - Rational Root Theorem

All real solutions of $x^3 - 8x^2 + 11x + 20 = 0$

$Q \to x^3$, which is just 1 as coefficient

$P \to$ factors of 20

1. $\boxed{P/Q}$ $x = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{4}{1}, \pm\frac{5}{1}, \pm\frac{10}{1}, \pm\frac{20}{1}$

2. Test ± 1

1	1	-8	11	20
		1	-7	4
	1	-7	4	24

↑ remainder, so f(1) not a factor.

-1	1	-8	11	20
		-1	9	-20
	1	-9	20	0

FACTOR → $(x^2 - 9x + 20)(x+1)$. ↑ no remainder! So a factor.

3. $(x^2 - 9x + 20)(x+1)$

$(x-4)(x-5)(x+1)$ Solutions are: $x = 4, x = 5, x = -1$

1. List all the possible rational solutions
2. Test possible solutions using synthetic division, until a solution is found
3. Factor completely using the result of synthetic division.

Example #4 - Using Zeros to Write Polynomial Function

Write a function with the zeros 3 and $2+\sqrt{5}$ (due to the Irrational Conjugate Th... we know that $2-\sqrt{5}$ also a zero)

1. $f(x) = (x-3)(x-(2+\sqrt{5}))(x-(2-\sqrt{5}))$
2. $f(x) = (x-3)[(x-2)+\sqrt{5}][(x-2)-\sqrt{5}]$ $-\sqrt{5} \times +\sqrt{5} = -5$
3. $f(x) = (x-3)[(x-2)^2 - 5]$

   $f(x) = (x-3)(x^2 - 4x + 4 - 5)$

   $f(x) = (x-3)(x^2 - 4x - 1)$
4. $x^3 - 4x^2 - x - 3x^2 + 12x + 3$
5. $x^3 - 7x^2 + 11x + 3$ → thats the answer $\boxed{f(x) = x^3 - 7x + 11x + 3}$

1. Write f(x) in factored form
2. Regroup terms
3. Simplify & multiply
4. Multiply
5. Combine like terms

Answer Key

1. $\dfrac{a^3-4a^2+4a=0}{a}$

$a(a^2-4a+4)$

$a(a-2)(a-2)$

$a=0 \quad a=2 \quad a=2$ $\boxed{a=2 \text{ to the multiplicity of } 2,\ a=0}$

2. $5w^3=50w$

$-50w \quad -50w$

$5w^3-50w=0$

$5w(w^2-10)=0$ $\boxed{w=0 \quad w=\pm\sqrt{10}}$

$\dfrac{5w=0}{5 \quad 5}$ $\quad \sqrt{w^2}=\sqrt{10}$

$w=0 \quad w=\pm\sqrt{10}$

3. $v^3-2v^2-16v=-32=0$

$+32$

$(v^3-2v^2-16v+32)$

$v^2(v-2)-16(v-2)$

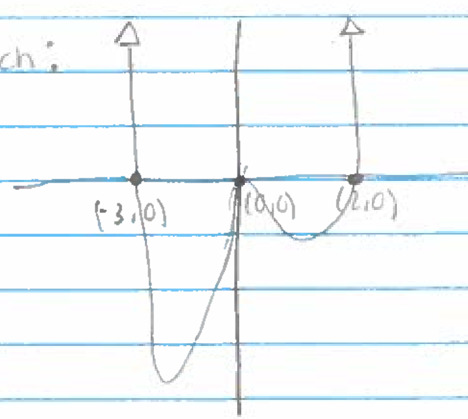
$(v^2-16)(v-2)$

$(v+4)(v-4)(v-2)$

$\boxed{v=-4 \quad v=4 \quad v=2}$

4. $\dfrac{x^4+x^3-6x^2=0}{x^2}$

Graph Sketch:



$x^2(x^2+x-6)$

$x^2(x+3)(x-2)$

$\boxed{x=0 \quad x=-3 \quad x=2 \rightarrow \text{the zeros}}$

Answer Key

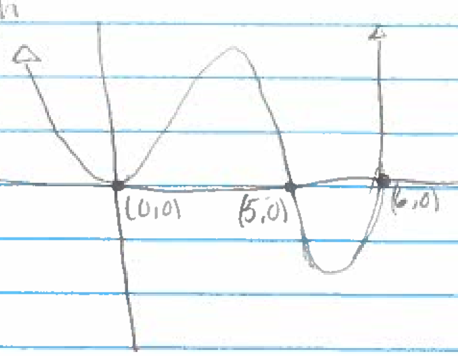
5. $\dfrac{x^6 - 11x^5 + 30x^4 = 0}{x^4}$

$x^4(x^2 - 11x + 30)$

$x^4(x-5)(x-6)$

$\boxed{x=0 \quad x=5 \quad x=6 \quad \text{zeros}}$

Sketch graph



6. $x^3 - 2x - 5x + 6 = 0$

$P/q = \pm 1 \quad \pm 2 \quad \pm 3 \quad \pm 6$

$1\,|\;\; 1 \quad -2 \quad -5 \quad 6 \qquad (x-1)$

$\qquad\quad 1 \quad -1 \quad -6 \qquad x=1$ is a solution

$\qquad 1 \quad -1 \quad -6 \quad 0$

$x^2 - x - 6 = 0$

$(x-3)(x+2)(x-1)$

$\boxed{x=3 \quad x=-2 \quad x=1 \quad \text{solutions}}$

7. $x^3 + 4x^2 - 11x - 30 = 0$

$P/q = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{5}{1}, \pm\frac{6}{1}, \pm\frac{10}{1}, \pm\frac{15}{1}, \pm\frac{30}{1}$

$3\,|\;\; 1 \quad 4 \quad -11 \quad -30 \qquad (x-3)$

$\qquad\quad 3 \quad 21 \quad 30 \qquad x=3$ is a solution

$\qquad 1 \quad 7 \quad 10 \quad 0$

$x^2 + 7x + 10$

$(x+5)(x+2)(x-3)$

$\boxed{x=-5 \quad x=-2 \quad x=3 \quad \text{solutions}}$

Answer Key

8. $2x^3 - 3x^2 - 50x - 24 = 0$

$p/q = \pm\frac{1}{1}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{4}{1}, \pm\frac{6}{1}, \pm\frac{8}{1}, \pm\frac{12}{1}, \pm\frac{24}{1}, \pm\frac{1}{2}, \pm\frac{3}{2},$

6	2	-3	-50	-24
		12	54	24
	2	9	4	0

$(x-6)$ → 6 is a solution

$2x^2 + 9x + 4$

-4	2	9	4
		-8	-4
	2	1	0

$(x+4)$ → -4 is a solution

$2x + 1 = 0$

$-1 \quad -1$

$\frac{2x}{2} = \frac{-1}{2}$

$\boxed{x = -1/2 \quad x = -4 \quad x = 6}$

9. $-2, 3, 6$

$(x+2)(x-3)(x-6)$

$(x^2 - x - 6)(x-6)$

$x^3 - 6x^2 - x^2 \cancel{+6x} \cancel{-6x} + 36$

$\boxed{f(x) = x^3 - 7x + 36}$

10. $4,\ 6-\sqrt{7}$, so → $6+\sqrt{7}$ also a factor due to Irrational Conjugates Thrm

$(x-4)(x-(6-\sqrt{7}))(x-(6+\sqrt{7}))$

$(x-4)\ (x-6)(x-6) - (\sqrt{7})^2$

$(x-4)(x-6)^2 - 7$

$(x-4)(x^2 - 12x + 36 - 7)$

$(x-4)(x^2 - 12x + 29)$

$x^3 - 12x^2 + 29x - 4x^2 + 48x - 116$

$\boxed{f(x) = x^3 - 16x^2 + 77x - 116}$

Answer Key

11. $4, 1-\sqrt{5}$

$\downarrow$

so $1+\sqrt{5}$ also a factor due to Irrational Conjugates Thrm

$(x-4)(x-(1-\sqrt{5}))(x-(1+\sqrt{5}))$

$(x-4)(x-1)(x-1)-(\sqrt{5})^2$

$(x-4)(x-1)^2-5$

$(x-4)(x^2-2x+1-5)$

$(x-4)(x^2-2x-4)$

$x^3-2x^2-4x-4x^2+8x+16$

$\boxed{f(x)=x^3-6x^2+4x+16}$